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Lagrangian structures, integrability and chaos for 3D dynamical equations

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Abstract
In this paper, we consider the general setting for constructing action principles for three-dimensional first-order autonomous equations. We present the results for some integrable and non-integrable cases of the Lotka–Volterra equation, and show Lagrangian descriptions which are valid for systems satisfying Shil'nikov criteria on the existence of strange attractors, though chaotic behaviour has not been verified up to now. The Euler–Lagrange equations we get for these systems usually present 'time reparametrization' invariance, though other kinds of invariance may be found according to the kernel of the associated symplectic 2-form. The formulation of a Hamiltonian structure (Poisson brackets and Hamiltonians) for these systems from the Lagrangian viewpoint leads to a method of finding new constants of the motion starting from known ones, which is applied to some systems found in the literature known to possess a constant of the motion, to find the other and thus showing their integrability. In particular, we show that the so-called ABC system is completely integrable if it possesses one constant of the motion.

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1. Introduction

Three-dimensional autonomous first-order nonlinear systems show a very rich behaviour, from complete integrability [1–4] to chaos and strange attractors [5–9], according to the values of the parameters that appear in the equations of motion. As they usually represent physical and biological systems of interest (Lorenz equation for hydrodynamic flow, Rössler equation for chemical reactive systems, Lotka–Volterra equation for laser physics, plasma physics [10], biology [11, 12], economy [13], etc), the lack of action principles for dealing with these equations is somewhat surprising. We intend to fill this gap using the general setting for Lagrangian structures [14], which when applied to three-dimensional systems reduces to a simpler problem, namely: for a given three-dimensional vector field V^a, to find the determinant of a metric (or volume element) such that the covariant divergence $\nabla_a V^a = 0$, which is a way of writing Liouville's theorem and also a condition which is known to lead to the construction of constants of the motion starting from symmetries only [15]. The Euler–Lagrange equations that come out from these principles are usually equivalent to $\frac{\mathrm{d}x^a}{\mathrm{d}s} = V^a$, where s is an arbitrary parameter (time reparametrization symmetry).

In section 2, a preview of known and new results on the Lagrangian approach for three-dimensional systems is presented. Section 3 deals with applications of the theory: a method to construct new constants of the motion from known ones, construction of Hamiltonian structures and several kinds of action principles with invariance related to symmetries. In section 4, we show examples for the Lotka–Volterra equation, including integrable cases as well as non-integrable ones: we show that recently found 'quasi-integrable' cases [3] (i.e., systems possessing one constant of the motion) of Lotka–Volterra are indeed integrable; for integrable cases, the construction of Lagrangians leads to new related equations, involving symmetries (of the evolution vector V) which generate the kernel of the symplectic 2-form. The Euler–Lagrange equations that arise possess several types of invariance, the simplest of which is time reparametrization, which arises when the kernel of the symplectic 2-form is generated by the evolution vector V; finally, two examples (related to known chaotic systems in the literature [5, 6]) show that Lagrangian descriptions for systems near chaotic behaviour do exist and they satisfy Shil'nikov conditions [7] on the existence of strange attractors around the fixed points, though it is still uncertain whether a homoclinic curve exists or not for such systems.

2. Preview

Consider the autonomous equations of motion

$$\dot{x}^a(t) = V^a[x^b(t)] \qquad a, b = 1, 2, 3 \tag{1}$$

where V defines a *flux*, velocity or evolution vector field in the corresponding space, being a function (which is local in time) of the *coordinates* $x^a(t)$. As an example, the Lorenz equation is defined by the vector

$$V[x, y, z] \equiv \{V^x, V^y, V^z\} = \{-\sigma x + \sigma y, rx - y - xz, xy - bz\}$$

where (σ, r, b) are three independent parameters.

2.1. Lagrangian structures for three-dimensional systems

In arbitrary dimensions, the problem of finding Lagrangian formulations or action principles for the given equations of motion (1) leads to covariant equations [14] in terms of Lie derivatives of tensor structures along the flux vector. In three dimensions, the equations are very restricted: as it is an odd-dimensional system, the Euler–Lagrange equations always have an invariance. For example, time reparametrization invariance appears when the kernel of the symplectic 2-form is generated by the evolution vector field V.

From now on, unless explicitly stated, we will be dealing with objects which are *time-independent*, in the sense that they only depend on time implicitly, through their dependence on the coordinates $x^a(t)$.

The equations of motion (1) are related to a *variational principle* with *action*

$$S[x^a(t), t] = \int_{t_0}^{t_1} \mathrm{d}t\,(L_a(\dot{x}^a - V^a) + K) \qquad (2)$$

(we assume from here on a summation over repeated indices), where the 1-form $L[x^b]$ and the 0-form $K[x^b]$ satisfy the following equation:

$$L_{a,b}V^b + L_b V^b_{,a} = K_{,a}$$

with $K_{,a} \equiv \frac{\partial K}{\partial x^a}$.

We rewrite the above equation in terms of invariant structures:

$$\underset{V}{\mathcal{L}}\, L = \mathrm{d}K$$

where $\underset{V}{\mathcal{L}}$ is the *Lie derivative* along the vector V, and d is the *exterior differential* (see [16] for a definition of these operators).

We call the pair $(L; K)$, 1-form L and 0-form K, a *standard Lagrangian pair* for V if $K \neq 0$. In the special case $K = 0$ we call L a non-standard Lagrangian for V: this case allows for the construction of Poisson brackets and constants of the motion (see section 3).

The *Euler–Lagrange* equations which come from the action (2) are

$$\Sigma_{ab}(\dot{x}^b - V^b) = 0 \qquad (3)$$

where $\Sigma \equiv \mathrm{d}L$ is the *symplectic 2-form* or Lagrange bracket whose components are

$$\Sigma_{ab} = L_{b,a} - L_{a,b}.$$

Note that these Euler–Lagrange equations do not imply the original equations of motion (1); instead they imply deformed or mixed equations involving the *kernel* of the symplectic 2-form. In three dimensions, the symplectic 2-form always has a kernel, and the simplest thing to do is to assume the kernel to be proportional to the evolution vector V itself (for other types of kernel see section 3.3):

$$\Sigma_{ab} = \rho[x^d]\epsilon_{abc}V^c$$

where ϵ_{abc} is the Levi-Civita 3-form, a totally antisymmetric tensorial density, with $\epsilon_{123} = 1$, and $\rho[x^a]$ is the *volume element*, a scalar density which solves [15]

$$(\rho V^a)_{,a} = 0 \qquad (4)$$

which is sufficient and necessary in order that Σ be a closed symplectic 2-form for the evolution vector V. In terms of a metric g_{ab}, the volume element is $\rho = \sqrt{\det g_{ab}}$ and the above equation is equivalent to $\nabla_a V^a = 0$, where the covariant derivative is taken with respect to the metric g. Geometrically, the existence of such a metric leads to the construction of an *integral invariant* [17] (Liouville theorem) of the form

$$I = \int_M \rho[x^a]\,\mathrm{d}^3x$$

where M is a region of the configuration space which is assumed to evolve along the flux V. The volume element is not unique: in fact, given two volume elements ρ, ρ', their quotient $C = \rho/\rho'$ is a constant of the motion.

As a consequence of the fact that the kernel of the symplectic 2-form is generated by the evolution vector, our action principles for three-dimensional systems will be purely geometrical: they will give only the curve of the motion, leaving the way local clocks run undetermined. The Euler–Lagrange equations (3) are thus $\Sigma_{ab}\dot{x}^b = 0$, which are equivalent to $\frac{\mathrm{d}x^a}{\mathrm{d}s} = V^a$, where s is an arbitrary parameter.

The standard Lagrangian pair $(L; K)$ is constructed (locally) by a line integral

$$\begin{aligned} K[x^b] &= L_a[x^b]V^a[x^b] \\ L_a[x^b] &= \int_0^1 \Sigma_{ab}[x^c(s)]\frac{\mathrm{d}x^b(s)}{\mathrm{d}s}s\,\mathrm{d}s + R_{,a}[x^b] \end{aligned} \tag{5}$$

where R is an arbitrary 0-form and the path in the line integral is parametrized by

$$x^a(s) = x_0{}^a + s(x^a - x_0{}^a), \qquad s \in [0, 1]$$

and is such that the integrand is equal to zero at $s = 0$ and well behaved along the path. The action for the system clearly shows time reparametrization invariance:

$$S[x^a(t)] = \int_{t_0}^{t_1} L_a[x^b(t)]\dot{x}^a(t)\,\mathrm{d}t.$$

It is important to mention here the closeness of the equation for the volume element with that for the *Darboux polynomial*. In the usual terminology [2], two polynomials in the coordinates $f(x^a)$ (the Darboux polynomial) and $Q(x^a)$ (the *cofactor*) are to be found in such a way that

$$f_{,a}V^a = Qf$$

therefore we may get a solution for the volume element, equation (4) from a Darboux polynomial when the cofactor is a multiple of $V^a{}_{,a}$. There are usually other solutions for the volume element such that the resulting expression for f is non-polynomial, as it happens with the so-called exponential factors in [3].

Anyway, this closeness may be used directly to construct action principles for every equation for which a Darboux polynomial or 'Darboux function' is known, as it happens in the Lorenz equation, the Rössler equation, the Lotka–Volterra equation and many others.

3. Applications

In the following applications, we assume that the evolution vector V possesses a non-standard Lagrangian L (this fixes R in equation (5)).

3.1. Method to construct a constant of the motion starting from a known one

We may use a known constant of the motion to construct the other explicitly. We will depict the method, omitting the details for simplicity.

Assume that the evolution vector V possesses: (a) a non-standard Lagrangian L, along with its associated volume element ρ; (b) a time-independent constant of the motion $H[x^a]$. Consider the vector defined by the components

$$\eta^a = \frac{1}{\rho}\epsilon^{abc}H_{,b}L_c.$$

It follows that η is a symmetry for V, and that η is proportional to V. Therefore we obtain

$$\eta^a = IV^a$$

where I is a constant of the motion for V, which may or may not be independent of H. We will show that the system is completely *integrable* (i.e., it possesses two independent, time-independent constants of the motion) in any case:

(i) if $I = 0$, then $DH_{,a} = L_a$ where D is an H-independent constant of the motion, because, otherwise, $\mathrm{d}L$ would vanish,

(ii) if $I \neq 0$ is independent of H, it is direct,

(iii) if $I \neq 0$ is a function of H, $I = I(H)$, define the 0-form $P(H)$ s.t. $\frac{\mathrm{d}P}{\mathrm{d}H} = \frac{1}{I(H)}$. Define next the 1-form $U = \mathrm{e}^{-P}L$. It follows that $\mathrm{d}U = 0$, therefore the 0-form C defined (locally) by

$$\mathrm{d}C = \mathrm{e}^{-P}L$$

is an H-independent constant of the motion for V.

3.2. Hamiltonian theories

Using the constants of the motion obtained in the last subsection, it is easy to construct all possible Hamiltonian theories for the evolution vector V. For example, for cases (ii) and (iii) above we may write

$$V^a = J^{ab} H_{,b}$$

where $J^{ab} = (I\rho)^{-1}\epsilon^{abc}L_c$ is the *Poisson bracket*. The *Casimir*, though always computable, is easy to guess only in case (iii): it is the above 0-form C, and it solves $J^{ab}C_{,b} = 0$. The *Hamiltonian* is H modulo a function of C; its final selection relies on stability conditions (see [18]).

3.3. More Lagrangian theories

If we have two independent constants of the motion for system (1), then another time-dependent constant of the motion may be constructed (in principle) by integration of any component of the evolution vector. If $H^1[x, y, z]$, $H^2[x, y, z]$ are two constants, assume we may solve for y and z, getting $y = y[x, H^1, H^2]$, $z = z[x, H^1, H^2]$. Thus the first component of the equations of motion, $\dot{x} = V^1[x, y, z]$, is transformed into $\dot{x} = F[x, H^1, H^2]$ and therefore we get

$$\widetilde{H}^3[x, y, z, t] = H^3[x, y, z] - t$$

where $H^3[x, y, z] = (\int^x (F[x, H^1, H^2])^{-1}\mathrm{d}x)\big|_{H^j = H^j[x,y,z]}$.

Now that we have three constants of the motion, we assert that the following is the generic form for any non-standard Lagrangian 1-form [14],

$$L = C^1\,\mathrm{d}C^2$$

where $C^j = C^j[H^1, H^2, H^3 - t]$, $\quad j = 1, 2$.

In the case that any C^j depends on H^3, the associated kernel of the symplectic 2-form is no longer proportional to V: indeed it is proportional to a (probably time-dependent) symmetry η for V, which may be calculated explicitly. We get, as a result, new Euler–Lagrange equations which mix the vector V with η. Examples with time-independent symmetries η will be shown in section 4.2.

3.4. Symmetries

Finally, assuming that the 0-forms H^1, H^2, H^3 from the last subsection are known, then an Abelian algebra of vectors $\mathcal{A} = \{\eta_1, \eta_2, \eta_3 \equiv V\}$ is constructed by taking the dual vectors of the 1-forms $\mathrm{d}H^1$, $\mathrm{d}H^2$, $\mathrm{d}H^3$, i.e., by solving for the vectors

$$\underset{\eta_j}{\mathcal{L}}\, H^k = \delta^k_j \qquad j, k = 1, 2, 3.$$

These commuting vectors are symmetries of each other by definition, and they are used to generate the kernel of the most general symplectic 2-form for the evolution vector V (see section 4.2).

4. Examples: the Lotka–Volterra equation

This equation, restricted to three dimensions, reads in its general form,

$$\begin{aligned}\dot{x} &= V^x[x, y, z] = x(a_1 + b_{11}x + b_{12}y + b_{13}z)\\ \dot{y} &= V^y[x, y, z] = y(a_2 + b_{21}x + b_{22}y + b_{23}z)\\ \dot{z} &= V^z[x, y, z] = z(a_3 + b_{31}x + b_{32}y + b_{33}z)\end{aligned}$$

where a_i, b_{ij} are constant parameters. We look for a volume element of the form

$$\rho[x, y, z] = x^u y^v z^w \tag{6}$$

where u, v, w are constants that depend on the above parameters.

For some values of the parameters, non-standard Lagrangian 1-forms may be found: their existence allows us to define Poisson brackets, and shows explicit integrability. In other cases, only standard Lagrangian pairs are obtained, though the question of whether these Lagrangians could be made non-standard by an addition of a closed 1-form is open. The examples suggest that, for chaotic or near chaotic systems, the answer to this question is negative.

4.1. Case $b_{ii} = 0, \quad i = 1, 2, 3$ (no Verhulst terms)

The evolution vector is, after rescaling,

$$V[x, y, z] = \{x(\lambda + Cy + z), y(\mu + Az + x), z(\nu + Bx + y)\} \tag{7}$$

where A, B, C, λ, μ and ν are constant parameters. Constants of motion for this system exist for a subset of the parameter space [1–4]. For arbitrary values of the parameters, however, this system is not integrable, but as we will see, a Lagrangian description always exists. Consider the volume element

$$\rho[x^a] = (xyz)^{-1}.$$

It is easy to see that $(\rho V^a)_{,a} = 0$, and the Lagrangian 1-form is obtained directly:

$$L_a[x^b] = (xyz)^{-1}\epsilon_{abc}(x^b V^c[x^d] + \eta^b W^c[x^d]) \tag{8}$$

where $\eta^a = M^a{}_b x^b$, $M^a{}_b = \mathrm{diag}(\lambda, \mu, \nu)$ and $W^a = x^a \ln x^a$.

Note that the above volume element is singular at the planes $x = 0, y = 0, z = 0$, reflecting the fact that these are invariant planes.

4.2. ABC system

When $\lambda = \mu = \nu = 0$, the system defined by the flow (7) is called the 'ABC system'. In this case, the Lagrangian (8) is non-standard, and reduces to

$$L_a[x^b] = (xyz)^{-1}\epsilon_{abc}x^b V^c[x^d].$$

The fact that this Lagrangian is non-standard leads us to conclude: if (for given values of the parameters) there is a constant of the motion, then another may be found using the theory of section 3. This may be applied, for example, to all the ABC systems studied in [4] where the author finds one polynomial constant of the motion. In the next examples, the method will be applied to some new cases found in [3].

As a final aside, the Lagrangian may be used to find Poisson brackets and Hamiltonian theories, and to find new, related equations of motion from the construction of a wide class of Lagrangian theories, as will be done in the last example of this subsection.

(i) Case $A = -1$, $B = 1/2$, $C = 0$.
This case possesses a recently found [3] constant of the motion: $H^1 = xy^{-1}z^2 \exp(-2(y+z)^2(xy)^{-1})$. We apply our Lagrangian to construct another constant of the motion. In the present case, it turns out that $U = (H^1)^{-1/2}L$ is a closed 1-form, which therefore must be the exterior derivative of some constant of the motion. We obtain, after integration,

$$H^2[x, y, z] = (H^1)^{-1/2}x - 2\int_0^{(y+z)/\sqrt{xy}} \exp(q^2)\,\mathrm{d}q$$

and thus this system is integrable.

(ii) Case $A = -1/(C+1)$, $B = 2$.
This is a new case also found in [3], with one constant of the motion: $H^1[x, y, z] = x^2|y|^{2(C+1)}|z|^{-2C}|2A^2xz - (y - Az)^2|^{C-1}$. According to our theory, in this case we obtain that the 1-form $U = (H^1)^{-\frac{1}{2(1+C)}}L$ is closed. We integrate and get the other constant of the motion:

$$H^2[x, y, z] = (y - Az)\left(1 - \Omega + xy\Omega^{AC}(1-\Omega)^{\frac{1}{2}+A}\frac{\Gamma(A)}{\Gamma(-AC)}\mathrm{B}_\Omega\left(-AC, \frac{1}{2} - A\right)\right)$$

where $\Omega = \frac{2A^2xz}{(y-Az)^2}$ and $\mathrm{B}_\Omega(a, b) \equiv \int_0^\Omega q^{a-1}(1-q)^{b-1}\,\mathrm{d}q$ is the incomplete beta function.

(iii) Case $ABC - 1 = 0$, $B(A+1) + 1 = 0$.
Here a quadratic constant of the motion [1, 3] is known: $H^1[x, y, z] = A^2(Bx - z)^2 - 2A(Bx + z)y + y^2$. We obtain the other constant

$$\begin{aligned} H^2[x, y, z] = {} & |x|^{-1}|yz|^{-1-C} \times |\pm\sqrt{H^1} + A(Bx - z) - y|^{1+2C} \\ & \times |(y + Az)x - (y - Az)C(\pm\sqrt{H^1} + y - Az)|. \end{aligned}$$

The above three systems are, therefore, integrable and the construction of many Hamiltonian as well as Lagrangian theories is possible. In the next example, we will take a known integrable case to show the construction of Lagrangian theories which give new equations of motion, in terms of the symmetries of the evolution vector field.

(iv) Case $ABC + 1 = 0$, $C = 1$.
Here, the integrability is guaranteed by the first condition [3]. We use the restriction $C = 1$ here for simplicity only. Known constants are $H^1[x, y, z] = -x + y - Az$ and $H^2[x, y, z] = xy^Bz^{-1}$. Now we take the y-component of the equation of motion (1), to obtain the third (time-dependent) constant of the motion:

$$\widetilde{H}^3[x, y, z, t] = H^3[x, y, z] - t$$

where

$$H^3[x, y, z] = -\frac{\ln\left(\frac{y}{x+Az}\right)}{H^1}.$$

The vectors η_1, η_2, η_3 dual to the 1-forms $\mathrm{d}H^1, \mathrm{d}H^2, \mathrm{d}H^3$ are commuting vectors, and they are found to be, after rearranging:

$$\begin{aligned} H^1\eta_1[x, y, z] &= \{x, y, z\} + H^3\eta_3[x, y, z] - BH^2\eta_2[x, y, z] \\ \eta_2[x, y, z] &= (x + Az)^{-1}y^{-B}z^2\{A, 0, -1\} \\ \eta_3[x, y, z] &= V[x, y, z] = \{x(y + z), y(x + Az), z(Bx + y)\}. \end{aligned}$$

It is clear from the definitions that any of the three equations of motion (labelled by j) which the above vectors define,

$$\dot{x}^a = \eta_j{}^a[x^b] \qquad j = 1, 2, 3$$

are completely integrable, with three constants of motion given by $C_j^k = H^k - t\delta_j^k, k = 1, 2, 3$.

Now we turn to the Lagrangian descriptions for the evolution vector $\eta_3 = V$ (the same could be done for the other vectors). Among all the possible examples, three special action principles are obtained with the Lagrangian 1-forms given by $L^1 = H^2\,\mathrm{d}H^3$, $L^2 = -H^1\,\mathrm{d}H^3$ and $L^3 = H^1\,\mathrm{d}H^2$. They are all non-standard Lagrangians for V, and the kernel of the symplectic matrix $\mathrm{d}L^j$ is easily found to be proportional to the vector η_j. Therefore, the Euler–Lagrange equations coming from the Lagrangian L^j are

$$\dot{x}^a = V^a + \alpha {\eta_j}^a$$

where α is an arbitrary 0-form. The case $j = 3$ gives the usual time reparametrization invariance, but the cases $j = 1, 2$ are examples of other kinds of invariance; their action principles are easily computed from equation (2).

4.3. Lagrangians and chaos

In the following examples we construct Lagrangians for Lotka–Volterra systems which are close to known chaotic systems. The general results here are: first, that a volume element of the form (6), implies ${V^a}_{,a} = 0$ at the finite *fixed point* (or singular point) of V which is not contained in any coordinate plane. This is somewhat surprising, because the volume element is singular at the coordinate planes, but its existence implies conditions on the vector field at a point which is far from the planes.

Second, for a large subset of the parameter space, these systems allow for Lagrangian descriptions along with Shil'nikov conditions on the existence of a strange attractor of spiral type [7] in the vicinity of the relevant fixed point, though no homoclinic curve has been found yet.

Third, the volume element becomes ill-defined (i.e., the powers u, v, w in equation (6) become infinite) when the parameter values are such that some fixed points of the vector field degenerate into a 'fixed line' (i.e., a line in the configuration space for which the evolution is frozen) and the system gets a constant of the motion, which may be calculated explicitly.

(i) A replicator-like equation. This example may be understood in the context of catalyst replication and mutation:

$$\begin{aligned} \dot{x} &= V^x[x, y, z] = x\left(\tfrac{1}{2}(1-x) + \tfrac{1}{2}(1-y) + \tfrac{1}{10}(1-z)\right) \\ \dot{y} &= V^y[x, y, z] = y\left(-\tfrac{1}{2}(1-x) - \tfrac{1}{10}(1-y) + \tfrac{1}{10}(1-z)\right) \\ \dot{z} &= V^z[x, y, z] = z\left(\lambda x + \mu(1-x) + \tfrac{1}{10}(1-y) + \tfrac{1}{10}(1-z)\right) \end{aligned} \tag{9}$$

where λ is a real parameter, and $\mu = -\frac{1}{6} - \lambda$. The volume element takes the form

$$\rho = x^{-\frac{1+6\lambda}{6\lambda}} y^{-\frac{1+6\lambda}{3\lambda}} z^{\frac{1-2\lambda}{2\lambda}}.$$

It is shown that $(\rho V^a)_{,a} = 0$. The Lagrangian 1-form is found to be

$$\begin{aligned} \{L_x, L_y, L_z\} = \rho\{&(1 + 5x - 3y - 3z)yz + (-30 + 60x - 9y)yz\lambda, \\ &0, 3xy(-11 + 5x + 5y + z) + 45xy^2\lambda\}. \end{aligned} \tag{10}$$

We note that the case $\lambda = 0$ and μ arbitrary is found in the literature, displaying a one-parameter family of strange attractors [6]. A bifurcation diagram in terms of μ may be found in [9]. We mention it because our case intersects with the latter at the point $\lambda = 0, \mu = -\frac{1}{6}$, where the volume element associated with the Lagrangian explodes. This point in the parameter space also represents a degeneration of some fixed points of the vector field, into the fixed line $\left\{1+\frac{3s}{10}, 1-\frac{s}{2}, 1+s\right\}$, $s \in \mathbb{R}$, and the following turns out to be a constant of the motion for the system (9):

$$C[x, y, z] = |x||y|^2|z|^{-3}.$$

(ii) A one predator–two prey system. We consider one of the most important models of predation [5] using the Lotka–Volterra equation, namely the equation

$$\begin{aligned}
\dot{x} &= V^x[x, y, z] = x\left(r - \frac{r}{K}x - \frac{r}{K}y - bz\right)\\
\dot{y} &= V^y[x, y, z] = y\left(r - \frac{r}{K}\alpha x - \frac{r}{K}y - (b-\epsilon)z\right)\\
\dot{z} &= V^z[x, y, z] = z(cbx + c(b-\epsilon)y - d).
\end{aligned}$$

The values of the parameters allow us to describe competitive superiority of prey x over prey y (case $\alpha > 1$), and predator-avoidance advantage of prey y over prey x (case $0 \leqslant \epsilon \leqslant b$). Now, if we keep the parameters arbitrary, we get the condition for the existence of the volume element

$$d = \frac{cK\epsilon^2}{b(\alpha - 1)} \tag{11}$$

and the volume element is

$$\rho[x, y, z] = \frac{1}{xyz}\left(x^{-\frac{\epsilon}{b}} y^{\frac{\epsilon}{b-\epsilon}} z^{\frac{r(\alpha-1)}{cK(b-\epsilon)}}\right)^{\frac{1}{\alpha-\alpha_0}}$$

where $\alpha_0 \equiv \frac{b}{b-\epsilon} - \frac{\epsilon}{b}$.

In [5], the author finds a set of values of the parameters for which the system develops spiral chaos [7, 8]. The values are $\alpha = 1.5$, $b = 0.01$, $c = 0.5$, $d = 1$, $r = 1$, $K = 1000$ and $\epsilon = 0.009$.

It is easy to check that there is no volume element of the form (6) for the above set of values, because condition (11) is not met. If we keep the values of the other parameters, the volume element exists for $d = 8.1$, which is much larger than the older value, $d = 1$. According to the model, this means that the predator (z variable) has a larger mortality rate when there is a volume element of the form (6) than in the chaotic case.

On the other hand, a study of this system under condition (11) shows that, at the fixed point $(x_0, y_0, z_0) \in \mathbb{R}_- \times \mathbb{R}^2_+$, Shil'nikov criteria on the existence of a strange attractor of spiral type [7] are met, for a large subset of the parameter space. However, no homoclinic curve or chaotic behaviour has been found numerically yet. The volume element becomes ill-defined when $\alpha = \alpha_0$: again, some fixed points of V degenerate into a fixed line and the system gets the following constant of the motion:

$$C[x, y, z] = |x|^{-\frac{d}{r\epsilon}(b-\epsilon)}|y|^{\frac{d}{r\epsilon}b}|z|.$$

5. Conclusions

Lagrangian descriptions for three-dimensional systems are directly related to the existence of determinants of metrics such that the covariant divergence of the evolution vector is zero, which in turn implies that there is an invariant volume element for the system. As a feature of odd-dimensional systems, the Euler–Lagrange equations we get usually possess time reparametrization invariance. A variational principle for three-dimensional evolution equations may thus be useful for the study of (quasi-)periodic orbits and long-time properties, or simply to test numerical results.

In the case of the Lotka–Volterra equation, the very existence of the volume element implies some condition on the vector field at the fixed point which is relevant for the chaotic attractor. This condition is compatible with Shil'nikov criteria for the existence of spiral chaos, but we have not seen chaotic behaviour numerically. Anyway, the condition does not imply at all the integrability of the system. On the contrary, the examples show that the volume element becomes ill-defined when constants of the motion appear through degeneracy of fixed points of the flow into fixed lines.

Recently found quasi-integrable systems (e.g., ABC systems with one constant of the motion) are shown to be integrable using a newly devised method to find a constant of the motion starting from a known one. Finally, the Lagrangian viewpoint for integrable systems leads to Euler–Lagrange equations with several kinds of invariance (including time reparametrization), according to the kernel of the associated symplectic 2-form.

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